Semiorthogonal decomposition via categorical representation theory

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Joint work with Yu Zhao (IPMU) in progress

NCTS

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Remark

In this talk, we work over the field $\mathbb{C}$ of complex numbers.

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We can consider a more general case, which is the representation of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. The third data is replaced by $(ef - fe)|_{V_\lambda} = [\lambda]_q Id_{V_\lambda}$, where $[\lambda]_q := q^{\lambda-1} + q^{\lambda-3} + ... + q^{-\lambda+1}$ is the quantum integer.

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- Geometry is a good resource for producing categories.
- It can be decategorified to recover the original vector space.

$$\text{vector spaces} \underset{decategorify}{\overset{categorify}{\rightleftarrows}} \text{categories}$$

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Let $\mathcal{D}^b Con(\mathbb{G}(k,N))$ to be the bounded derived categories of constructible sheaves on $\mathbb{G}(k,N)$. These will be our weight categories $\mathcal{K}(\lambda) = \mathcal{D}^b Con(\mathbb{G}(k,N))$, where $\lambda = N - 2k$.

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Here $p_1 : Fl(k-1,k) \to \mathbb{G}(k,N)$ and $p_2 : Fl(k-1,k) \to \mathbb{G}(k-1,N)$ are natural projections.

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Here $p_1 : Fl(k-1,k) \to \mathbb{G}(k,N)$ and $p_2 : Fl(k-1,k) \to \mathbb{G}(k-1,N)$ are natural projections. We define the following functors

$$\mathsf{E} := p_{2*}p_1^* : \mathcal{D}^b Con(\mathbb{G}(k,N)) = \mathcal{K}(\lambda) \to \mathcal{D}^b Con(\mathbb{G}(k-1,N)) = \mathcal{K}(\lambda+2)$$

$$\mathsf{F} := p_{1*}p_2^* : \mathcal{D}^b Con(\mathbb{G}(k-1,N)) = \mathcal{K}(\lambda+2) \to \mathcal{D}^b Con(\mathbb{G}(k,N)) = \mathcal{K}(\lambda)$$

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Theorem 1 (Beilinson-Lusztig-MacPherson, Chuang-Rouquier)

The categories and functors defined above gives a categorical $\mathfrak{sl}_2$ action. This means that the functors defined above satisfy

$$\mathsf{EF}|_{\mathcal{K}(\lambda)} \cong \mathsf{FE}|_{\mathcal{K}(\lambda)} \bigoplus \mathsf{Id}_{\mathcal{K}(\lambda)}^{\oplus \lambda} \text{ if } \lambda \geq 0$$

$$\mathsf{FE}|_{\mathcal{K}(\lambda)} \cong \mathsf{EF}|_{\mathcal{K}(\lambda)} \bigoplus \mathsf{Id}_{\mathcal{K}(\lambda)}^{\oplus -\lambda} \text{ if } \lambda \leq 0$$

The motivation of our problem

Motivated by the above result, we replace constructible sheaves with coherent sheaves. Roughly speaking, this means that instead of studying locally constant functions (constructible sheaves), we study holomorphic functions (coherent sheaves).

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Thus our weight categories $\mathcal{K}(\lambda)$ are the bounded derived categories of coherent sheaves on $\mathbb{G}(k, N)$, which is denoted by $\mathcal{D}^b Coh(\mathbb{G}(k, N))$, where $\lambda = N - 2k$.

Our functors

$$Fl(k-1,k) = \{0 \overset{k-1}{\subset} V' \overset{1}{\subset} V \overset{N-k}{\subset} \mathbb{C}^N\}$$

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Denoting $\mathcal{V}, \mathcal{V}'$ to be the tautological bundles on $Fl(k-1,k)$ of rank k, $k-1$ respectively, then there is a natural line bundle $\mathcal{V}/\mathcal{V}'$ on $Fl(k-1,k)$.

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Denoting $\mathcal{V}, \mathcal{V}'$ to be the tautological bundles on $Fl(k-1,k)$ of rank k, $k-1$ respectively, then there is a natural line bundle $\mathcal{V}/\mathcal{V}'$ on $Fl(k-1,k)$. Instead of just pullback and pushforward, we have more functors

$$\mathsf{E}_r := p_{2*}(p_1^* \otimes (\mathcal{V}/\mathcal{V}')^r) : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k-1,N))$$
$$\mathsf{F}_r := p_{1*}(p_2^* \otimes (\mathcal{V}/\mathcal{V}')^r) : \mathcal{D}^b Coh(\mathbb{G}(k-1,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k,N))$$

where $r \in \mathbb{Z}$.

The main problem

Problem.
We want to understand how this $L\mathfrak{sl}_2 := \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$-like algebra acting on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k, N))$, where $e \otimes t^r$ and $f \otimes t^s$ acting via the functors E_r and F_s respectively for $r,\ s \in \mathbb{Z}$.

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1. What are the categorical commutator relations between $\mathsf{E}_r \mathsf{F}_s$ and $\mathsf{F}_s \mathsf{E}_r$?
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3. If we define the algebra, can we give a definition of its categorical action like $\mathfrak{sl}_2$ in the introduction?

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Theorem 2 (Hsu)

(1)The resulting algebra acting on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$ is a new algebra, which we call it the shifted $q=0$ affine algebra. Denoted by $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$.

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(2)We give a definition of the categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action.
(3)We verify that there is a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$.

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Remark

More generally, we constructed a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$ action on the derived categories of coherent sheaves on n-step partial flag varieties.

Tool: Fourier-Mukai (FM) transforms

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Definition 3

Let X, Y be two smooth projective varieties. A Fourier-Mukai (FM) kernel is any object $\mathcal{P} \in \mathcal{D}^b Coh(X \times Y)$. For such $\mathcal{P}$ we define the associated Fourier-Mukai (FM) transform, which is the functor

$$\Phi_{\mathcal{P}} : \mathcal{D}^b Coh(X) \to \mathcal{D}^b Coh(Y)$$

$$\mathcal{F} \mapsto \pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{P})$$

where π_1, π_2 are natural projections.

FM kernels for E_r and F_s

Then the functor $\mathsf{E}_r : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k-1,N))$ isomorphic to a FM transform with the kernel

$$\mathcal{E}_r \mathbf{1}_{(k,N-k)} := \iota_*(\mathcal{V}/\mathcal{V}')^r \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k-1,N))$$

where $\iota : Fl(k-1,k) \to \mathbb{G}(k,N) \times \mathbb{G}(k-1,N)$ is the natural inclusion, i.e., $\mathsf{E}_r \cong \Phi_{\mathcal{E}_r \mathbf{1}_{(k,N-k)}}$.

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$$\mathcal{F}_s \mathbf{1}_{(k,N-k)} \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k+1,N))$$

to be the FM kernel for F_s, i.e., $\mathsf{F}_s \cong \Phi_{\mathcal{F}_r \mathbf{1}_{(k,N-k)}}$.

Categorical commutator relations between E_r and F_s

Then $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)},\ (\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k,N))$ are FM kernels for the functors $\mathsf{E}_r \circ \mathsf{F}_s,\ \mathsf{F}_s \circ \mathsf{E}_r$, respectively.

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Definition 4

An object $E \in \mathrm{Ob}(\mathcal{D})$ is called exceptional if

$$\mathrm{Hom}_{\mathcal{D}}(E, E[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{if } l \neq 0. \end{cases}$$

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Then we define the notion of exceptional collections.

Definition 5

An ordered collection $\{E_1, ..., E_n\}$, where $E_i \in \mathrm{Ob}(\mathcal{D})$ for all $1 \leq i \leq n$, is called an exceptional collection if each E_i is exceptional and moreover $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[l]) = 0$ for all $i > j$ and all $l \in \mathbb{Z}$.

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$\{E_1, ..., E_n\}$	$\chi(E_i, E_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$

A brief interpretation

We can interpret exceptional collections in the following rough dictionary.

category theory	linear algebra
$\mathcal{D}$	finite dimensional vector space $K_0(\mathcal{D})$
$\mathrm{Hom}_{\mathcal{D}}(-,-)$	Euler form $\chi = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(-,-[i])$
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Remark

The word "semi" comes from the fact that the Euler form χ is NON-symmetric.

Semiorthogonal decompositions

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Definition 6

A semiorthogonal decomposition (SOD for short) of $\mathcal{D}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, ..., \mathcal{A}_n$ such that

1. there is no non-zero Homs from right to left, i.e. $\mathrm{Hom}_{\mathcal{D}}(A_i, A_j) = 0$ for all $A_i \in \mathsf{Ob}(\mathcal{A}_i)$, $A_j \in \mathsf{Ob}(\mathcal{A}_j)$ where $1 \leq j < i \leq n$.
2. $\mathcal{D}$ is generated by $\mathcal{A}_1, ..., \mathcal{A}_n$, i.e. the smallest full triangulated subcategory containing $\mathcal{A}_1, ..., \mathcal{A}_n$ equal to $\mathcal{D}$.

We will use the notation $\mathcal{D} = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$ for a semiorthogonal decomposition of $\mathcal{D}$ with components $\mathcal{A}_1, ..., \mathcal{A}_n$.

SOD given by exceptional collection

Note that an exceptional collection of $\mathcal{D}$ naturally gives rise to a SOD of $\mathcal{D}$.

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Let $\{E_1, ..., E_n\}$ be an exceptional collection of $\mathcal{D}$. Then we have the following SOD

$$\mathcal{D} = \langle \mathcal{A}, E_1, ..., E_n \rangle.$$

where $\mathcal{A} = \langle E_1, ..., E_n \rangle^{\perp}$ and E_i denote the full triangulated subcategory generated by the object E_i.

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Remark

For a full triangulated subcategory $\mathcal{C} \subset \mathcal{D}$, we define $\mathcal{C}^{\perp} = \{X \in \mathsf{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(C, X) = 0 \ \forall \ C \in \mathsf{Ob}(\mathcal{C})\}$ to be the right orthogonal to $\mathcal{C}$ in $\mathcal{D}$. It is a triangulated subcategories of $\mathcal{D}$.

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Remark

An exceptional collection is called full if the subcategory $\mathcal{A}$ is zero.

The Beilinson-Kapranov exceptional collection

The simplest example is given by Beilinson for projective space $\mathbb{P}^{N-1} = \mathbb{G}(1, N)$.

Theorem 7 (Beilinson)

There is a full exceptional collection (thus a SOD)

$$\mathcal{D}^b Coh(\mathbb{P}^{N-1}) = \langle \mathcal{O}_{\mathbb{P}^{N-1}}(-N+1), \mathcal{O}_{\mathbb{P}^{N-1}}(-N+2), ..., \mathcal{O}_{\mathbb{P}^{N-1}} \rangle.$$

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Theorem 8 (M. Kapranov)

There is a full exceptional collection (thus a SOD*)*

$$\mathcal{D}^b Coh(\mathbb{G}(k, N)) = \langle\ \mathbb{S}_\lambda \mathcal{V}\ \rangle_{\lambda \in P(N-k,k)}.$$

Relate to the categorical action

Since we construct an action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$ via using FM kernels, we try to relate the Kapranov exceptional collection to this action.

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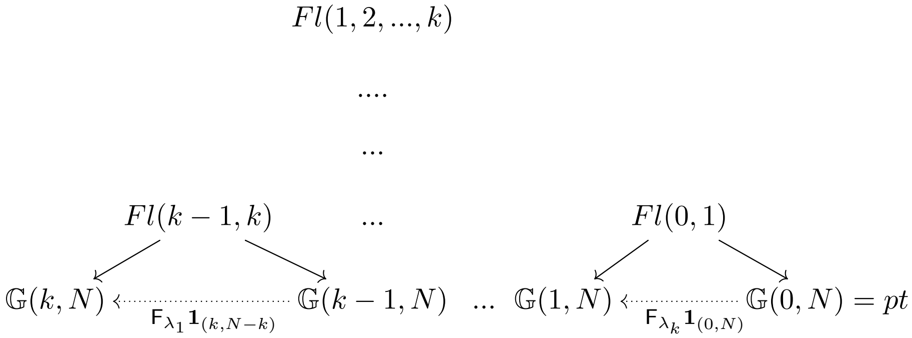

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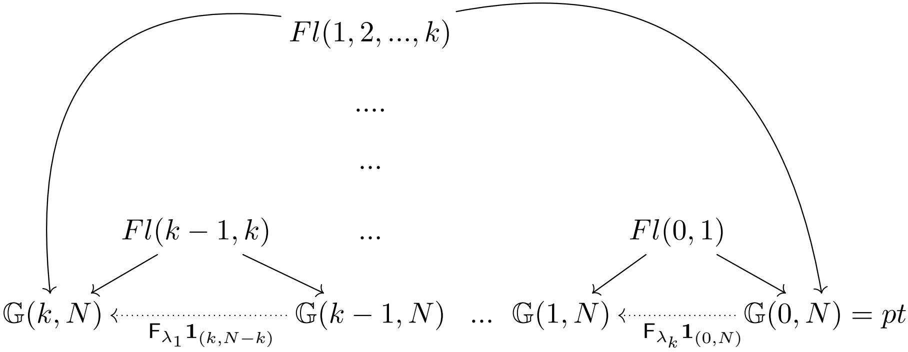

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More precisely, by using the Borel-Weil-Bott theorem we get

$$\mathbb{S}_\lambda \mathcal{V} \cong \mathcal{F}_{\lambda_1} * ... * \mathcal{F}_{\lambda_k} \mathbf{1}_{(0,N)}$$

where $\lambda = (\lambda_1, ..., \lambda_k) \in P(N-k,k)$. Note that $\mathcal{F}_{\lambda_1} * ... * \mathcal{F}_{\lambda_k} \mathbf{1}_{(0,N)}$ is the FM kernel for the functor $\mathsf{F}_\lambda \mathbf{1}_{(0,N)} := \mathsf{F}_{\lambda_1} ... \mathsf{F}_{\lambda_k} \mathbf{1}_{(0,N)}$.

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We know that $\{\mathbb{S}_\lambda \mathcal{V}\}_{\lambda \in P(N-k,k)}$ is an exceptional collection, it is natural to ask the following question.
$\boldsymbol{Question}$ **:** Given an (abstract) categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action on $\mathcal{K}$. Do the collection of functors

$$\{\mathsf{F}_\lambda \mathbf{1}_{(0,N)} : \mathcal{K}(0, N) \to \mathcal{K}(k, N-k)\}_{\lambda \in P(N-k,k)}$$

behave like an exceptional collection?

SOD of weight categories

Proposition 9 (Hsu)

The functors $\{\mathsf{F}_\lambda \mathbf{1}_{(0,N)}\}_{\lambda \in P(N-k,k)}$ *satisfy the following properties*

(1) $\mathrm{Hom}(\mathsf{F}_\lambda \mathbf{1}_{(0,N)}, \mathsf{F}_\lambda \mathbf{1}_{(0,N)}) \cong \mathrm{Hom}(\mathbf{1}_{(0,N)}, \mathbf{1}_{(0,N)})$ (*exceptional-like*)

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Remark

When the weight categories are $\mathcal{K}(k, N-k) = \mathcal{D}^b Coh(\mathbb{G}(k,N))$, we have $\mathrm{Hom}(\mathbf{1}_{(0,N)}, \mathbf{1}_{(0,N)}) \cong \mathbb{C}$. This recovers the exceptional collection $\{\mathbb{S}_\lambda \mathcal{V}\}$.

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Remark

The first property (1) also implies that the functors $\mathsf{F}_\lambda \mathbf{1}_{(0,N)} : \mathcal{K}(0,N) \to \mathcal{K}(k, N-k)$ are fully faithful for $\lambda \in P(N-k,k)$.

SOD of weight categories

Since $\mathsf{F}_\lambda \mathbf{1}_{(0,N)} : \mathcal{K}(0,N) \to \mathcal{K}(k,N-k)$ is fully faithful for $\lambda \in P(N-k,k)$, it gives an equivalence from $\mathcal{K}(0,N)$ to the subcategory of $\mathcal{K}(k,N-k)$ generated by its essential images. By abusing of notation, we still denote it by $\mathcal{K}(0,N)$.

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Theorem 10 (Hsu)

Given a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action $\mathcal{K}$. There is a SOD

$$\mathcal{K}(k, N-k) = \langle \mathcal{A}(k, N-k), \binom{N}{k} - copies\ of\ \mathcal{K}(0,N) \rangle$$

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Remark

In fact, we prove the above theorem for the $\mathfrak{sl}_n$ case.

Natural questions

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$\boldsymbol{Question1}$ **:** Is there a geometric example $\bigoplus_k \mathcal{D}^b Coh(X_{(k,N-k)})$ for categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action such that the orthogonal complement $\mathcal{A}(k,N-k) \neq 0$? (Note that when $X_{(k,N-k)} = \mathbb{G}(k,N)$, $\mathcal{A}(k,N-k) = 0$ for all k).

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$\boldsymbol{Question2}$ **:** If so, i.e. $\mathcal{D}^b Coh(X_{(k,N-k)}) = \langle \mathcal{A}(k, N-k), \ldots. \rangle$ with $\mathcal{A}(k, N-k) \neq 0$ for all k, then is $\bigoplus_k \mathcal{A}(k, N-k)$ a (categorical) sub-representation of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$?

Recent works

Theorem 11 (Jiang-Leung, 2019)

Let X be a smooth projective variety, $\mathcal{G}$ a coherent sheaf on X with homological dimension ≤ 1. This implies that $\mathcal{G}$ admits a resolution $\mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{G}$ with $\mathcal{E}^0$, $\mathcal{E}^{-1}$ locally free.

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$$\mathcal{D}^b Coh(\mathbb{P}(\mathcal{G})) = \langle \mathcal{D}^b Coh(\mathbb{P}(\mathcal{H}), \mathcal{D}^b Coh(X)(1), ..., \mathcal{D}^b Coh(X)(r) \rangle$$

where $\mathcal{H} := \mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X)$ and $r :=$ rk$\mathcal{E}^0 -$ rk$\mathcal{E}^{-1}$ is the rank for $\mathcal{G}$.

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Remark

When $\mathcal{G}$ is locally free, this result recover the projective bundle formula by Orlov. Moreover, Jiang-Leung prove the above result for X to be a regular scheme.

Current work: Relative Grassmannian

Theorem 12 (Y. Toda, 2021)

Let X and $\mathcal{G}$ be the same as in Theorem 11. Then there is a SOD *for $\mathcal{D}^b Coh(Gr(\mathcal{G}, d)))$ which extends the result by Jiang-Leung, where $Gr(\mathcal{G}, d)$ is the Grassmannian parametrizes rank d locally free quotient of $\mathcal{G}$.*

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Remark

If fact, $\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X))$ is a Springer-type resolution of the singular locus Sing$(\mathcal{G}) := \{x \in X \mid \text{rk}\mathcal{G}_x > r\}$, and $r := \text{rk}\mathcal{E}^0 - \text{rk}\mathcal{E}^{-1} = \text{rk}\mathcal{G}$.

Thank you for your attention.